

On the Combinatorics of the Universal Enveloping Algebra $\widehat{U}_h(\mathfrak{sl}_2)$

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Abstract

We study using combinatorial methods the structural coefficients of the formal homogeneous universal enveloping algebra $\widehat{U}_h(\mathfrak{sl}_2)$ of the special linear algebra \mathfrak{sl}_2 , over a field of characteristic zero. We provide explicit formulae for the product of generic elements in $\widehat{U}_h(\mathfrak{sl}_2)$, and construct combinatorial objects giving flesh to these formulae.

1 Introduction and Basic Notions

When studying an algebra A we usually approach it from a couple of complementary viewpoints. On the one hand we study A by given generators for it, generators for the relations among generators, generators for the relations among relations, and so on, i.e. we study A by analyzing a free resolution for it. This approach was pioneered by Hilbert. On the other hand we study A by finding a basis $\{e_i\}$ for A , and studying the structural coefficients c_{ij}^k defined by the equation $e_i e_j = \sum_k c_{ij}^k e_k$. Note that we do not demand that A be finite dimensional, but do insist that the sum above be finite; otherwise further topological considerations have to be imposed. For example, one says that A is a combinatorial (resp. integral, rational) algebra if it admits a combinatorial (resp. integral, rational) basis, i.e. a basis such that the associated structural coefficients are such that $c_{ij}^k \in \mathbb{N}$ (resp. $c_{ij}^k \in \mathbb{Z}$, $c_{ij}^k \in \mathbb{Q}$.) Analogously, for $k = \mathbb{R}$, we say that A is a probabilistic algebra, if it admits a basis such that the associated structural coefficients satisfy $c_{ij}^k \geq 0$ and $\sum_k c_{ij}^k = 1$. This approach, strongly promoted by Rota [23], has been undertaken by many combinatorialist, for example, in the study of combinatorial Hopf algebras [3, 13, 22, 25].

We assume the reader to be familiar with the rudiments of the theory of Lie algebras [1, 6, 10, 11, 12, 14, 15, 19]. Fix a field k of characteristic zero. Given a Lie algebra L over k , we let $T(L)$ be the free k -algebra generated by L , and $U(L) = T(L)/I(L)$ be the universal enveloping algebra of L , where $I(L)$ is the two-sided ideal of $T(L)$ generated by elements of the form $xy - yx - [x, y]$ for $x, y \in L$.

Consider the Lie algebra $M_{2 \times 2}(k)$ of square matrices of size two with entries in k , and its Lie sub-algebra $\mathfrak{sl}_2 = \{A \in M_{2 \times 2}(k) \mid \text{tr}(A) = 0\}$ consisting of trace zero matrices. The universal enveloping algebra $U(\mathfrak{sl}_2)$ is given by

$$U(\mathfrak{sl}_2) = T(\mathfrak{sl}_2)/I(\mathfrak{sl}_2) = \frac{k \langle x, y, z \rangle}{I},$$

i.e. $U(\mathfrak{sl}_2)$ is the quotient of the free algebra $k \langle x, y, z \rangle$ generated by x, y and z by the ideal I generated by the identities:

$$yx = xy + 2x, \quad zx = xz - y, \quad zy = yz + 2z.$$

Pick a variable h algebraically independent from x, y, z . The homogeneous universal enveloping algebra $U_h(\mathfrak{sl}_2)$ is given by

$$U_h(\mathfrak{sl}_2) = \frac{k \langle x, y, z \rangle [h]}{I_h},$$

i.e. $U_h(\mathfrak{sl}_2)$ is the free associative k -algebra generated by the letters x, y, z and h , divided by the relations:

$$yx = xy + 2xh, \quad zx = xz - yh, \quad zy = yz + 2zh,$$

and h commutes with x, y, z . The methods employed in this work apply for homogeneous universal enveloping algebra $U_h(\mathfrak{sl}_2)$, nevertheless we are going to develop our results for the formal homogeneous universal enveloping algebra $\widehat{U}_h(\mathfrak{sl}_2)$ given by

$$\widehat{U}_h(\mathfrak{sl}_2) = \frac{k \langle\langle x, y, z \rangle\rangle [[h]]}{I_h},$$

i.e. we allow formal power series in the variables x, y, z, h subject to the same relations generating the ideal I_h from above.

Fix the order $x < y < z < h$ on the formal generators of $\widehat{U}_h(\mathfrak{sl}_2)$. A monomial in x, y, z, h is in normal form if it looks like $x^a y^b z^c h^d$. By the Poincaré-Birkhoff-Witt theorem [10, 14] all elements of $\widehat{U}_h(\mathfrak{sl}_2)$ can be written in a unique way as a formal sum of monomials in normal form, i.e. a generic element of $\widehat{U}_h(\mathfrak{sl}_2)$ can be written as

$$f = \sum_{a,b,c,d \in \mathbb{N}} f_{a,b,c,d} \frac{x^a y^b z^c h^d}{a!b!c!d!}.$$

Note that we use divided power monomials, i.e. monomials divided by the corresponding factorials. This is the main reason why we work with a characteristic zero field. The basis of divided power monomials is a priori a rational basis; a main result of this work is to show that it is actually an integral basis, i.e. we show that the product of formal

power series in normal form with divided power monomials can be written in normal form with divided power monomials through a sophisticated procedure, which we are going to dissect first from an algebraic viewpoint, and then adopting a combinatorial viewpoint. The reader may take a look at Figure 4 which, as we shall argue, summarizes the combinatorics behind $\widehat{U}_h(\mathfrak{sl}_2)$ arising from the basis of divided power monomials written in the prescribed order. Working with divided powers makes our combinatorial constructions more transparent, nevertheless we stress that they can be adapted to work with undivided monomials as well.

2 Explicit Formulae for the Product on $\widehat{U}_h(\mathfrak{sl}_2)$

In this section we provide explicit formulae for the product on $\widehat{U}_h(\mathfrak{sl}_2)$. Before attacking our problem we first introduced the required notation.

For $n \in \mathbb{N}$ we set $[n] = \{1, \dots, n\}$ for $n \geq 1$, and $[0] = \emptyset$. If x is a finite set we let $|x|$ be its cardinality and set $[z] = [|z|]$.

For $s, n \in \mathbb{N}$ with $0 \leq s \leq n$, we let the elementary symmetric function e_n^s on the variables x_1, \dots, x_n be given by

$$e_n^s(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \dots < i_s \leq n} x_{i_1} \dots x_{i_s} = \sum_{\substack{A \subseteq [n] \\ |A|=s}} \prod_{i \in A} x_i.$$

For $a \in \mathbb{N}$ and $0 \leq s \leq n$, we set $(a)_n^s = e_n^s(a, a+1, \dots, a+n-1)$. For $1 \leq s \leq n$, the symbol $(a)_n^s$ satisfies the recursion:

$$(a)_n^s = (a)_{n-1}^s + (a+n-1)(a)_{n-1}^{s-1},$$

as well as the boundary conditions

$$(a)_n^s = 0, \quad \text{for } s > n, \quad \text{and} \quad (a)_n^0 = 1, \quad \text{for } n \geq 0.$$

For example, for $n = 4$ and $s = 3$, we have that

$$(a)_4^3 = e_4^3(a, a+1, a+2, a+3) = 4a^3 + 18a^2 + 22a + 6.$$

A combinatorial interpretation for $(a)_n^s$ in terms of tableaux goes as follows: $(a)_n^s$ counts the number of ways of inserting s unlabeled dots into a tableaux with rows of length $(a+n-1, a+n-2, \dots, a+1, a)$, with at most one dot in each row. Figure 2 shows an example of a configuration contributing to the computation of $(2)_5^3$.

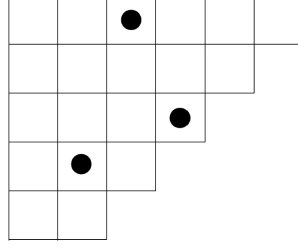


Figure 1: Configuration contributing to $(2)_5^3$.

In the proof of Proposition 2 below we use the Pochhammer k -symbol [7, 24] which is given for a, k in some algebra A and $n \in \mathbb{N}_{>0}$ by

$$(a)_{n,k} = a(a+k)(a+2k) \dots (a+(n-1)k).$$

We let $(a)_n = a(a-1)(a-2) \dots (a-n+1)$ be the falling factorial for $a, n \in \mathbb{N}$. Note that $(a)_n = (a)_{n,-1}$.

We are also going to use the following identities.

Lemma 1. For a, h in an algebra A , and $n \in \mathbb{N}_{>0}$ we have that:

1. $(a)_{n,-h} = (a - (n-1)h)(a)_{n-1,-h}$.
2. $(a)_{n,-h} = a(a-h)_{n-1,-h}$.
3.
$$\left(y + (a - (n+1))h \right)_{n,-h} = \sum_{w=0}^n (a-2n)_n^w y^{n-w} h^w.$$

Proof. Properties 1 and 2 follow from definition. To show 3 note that

$$\left(y + (a - (n+1))h \right)_{n,-h} = \prod_{i=0}^{n-1} \left[y + \left(a - (n+i+1) \right) h \right].$$

Developing the latter product assume we choose $w \in [n]$ times a h -term and thus $n-w$ times a y -term, this choice gives rise just to the sum $\sum_{w=0}^n$ and the factor $y^{n-w} h^w$ in our proposed formula. In order to get the factor h^w we must have chosen and multiplied w elements from the set

$$(a - (n+1)), (a - (n+2)), \dots, (a - (2n)),$$

which agrees, by definition, with the result of computing $(a-2n)_n^w$. \square

Next proposition forms the basis upon which all further results in this work are built. Related identities are given by Kac [18], and by Díaz and Pariguan [9].

Proposition 2. For $a, b \in \mathbb{N}$, the following identities hold in $\widehat{U}_h(\mathfrak{sl}_2)$:

$$\begin{aligned}
1. \quad \frac{z^a y^b}{a! b!} &= \sum_{k=0}^b (2a)^k \frac{y^{b-k}}{(b-k)!} \frac{z^a h^k}{a! k!}. \\
2. \quad \frac{y^a x^b}{a! b!} &= \sum_{k=0}^a (2b)^k \frac{x^b}{b!} \frac{y^{a-k}}{(a-k)!} \frac{h^k}{k!}. \\
3. \quad \frac{z^a x^b}{a! b!} &= \\
&\sum_{0 \leq w \leq v \leq \min(a, b)} (-1)^v (v-w)! (v+w)_w (a+b-2v)_v \frac{x^{b-v}}{(b-v)!} \frac{y^{v-w}}{(v-w)!} \frac{z^{a-v}}{(a-v)!} \frac{h^{v+w}}{(v+w)!}.
\end{aligned}$$

Proof. We show 1 and 3, item 2 follows from 1 using the automorphism of $\widehat{U}_h(\mathfrak{sl}_2)$ given on generators by

$$x \rightarrow z, \quad y \rightarrow -y, \quad z \rightarrow x, \quad \text{and} \quad h \rightarrow h.$$

We show item 1 by induction. First we check that

$$\frac{z^a}{a!} y = y \frac{z^a}{a!} + 2a \frac{z^a}{a!} h.$$

For $a = 1$ we get a defining identity for $\widehat{U}_h(\mathfrak{sl}_2)$. For $a > 0$ we have that:

$$\begin{aligned}
\frac{z^{a+1}}{(a+1)!} y &= \frac{z}{(a+1)} \left(\frac{z^a}{a!} y \right) = \frac{z}{(a+1)} \left(y \frac{z^a}{a!} + 2a \frac{z^a}{a!} h \right) = \\
\frac{1}{(a+1)} (yz + 2zh) \frac{z^a}{a!} + 2a \frac{z^{a+1}}{(a+1)!} h &= y \frac{z^{a+1}}{(a+1)!} + 2(a+1) \frac{z^{a+1}}{(a+1)!} h.
\end{aligned}$$

We proceed by induction on b . We have that:

$$\begin{aligned}
\frac{z^a}{a!} \frac{y^{b+1}}{(b+1)!} &= \left(\frac{z^a}{a!} \frac{y^b}{b!} \right) \frac{y}{(b+1)} = \sum_{k=0}^b \frac{(2a)^k}{(b+1)} \frac{y^{b-k}}{(b-k)!} \left(\frac{z^a}{a!} y \right) \frac{h^k}{k!} = \\
&\sum_{k=0}^b \frac{(2a)^k}{(b+1)} \frac{y^{b-k}}{(b-k)!} \left(y \frac{z^a}{a!} + (2a) \frac{z^a}{a!} h \right) \frac{h^k}{k!} = \\
&\sum_{k=0}^b \frac{(2a)^k}{(b+1)} \frac{y^{b+1-k}}{(b-k)!} \frac{z^a}{a!} \frac{h^k}{k!} + \sum_{k=0}^b \frac{(2a)^{k+1}}{(b+1)} \frac{y^{b-k}}{(b-k)!} \frac{z^a}{a!} \frac{h^{k+1}}{k!} =
\end{aligned}$$

$$\begin{aligned}
& \frac{y^{b+1}}{(b+1)!} \frac{z^a}{a!} + \sum_{k=1}^b \frac{(2a)^k}{(b+1)} \frac{y^{b+1-k}}{(b-k)!} \frac{z^a}{a!} \frac{h^k}{k!} + \\
& \sum_{k=0}^{b-1} \frac{(2a)^{k+1}}{(b+1)} \frac{y^{b-k}}{(b-k)!} \frac{z^a}{a!} \frac{h^{k+1}}{k!} + (2a)^{b+1} \frac{z^a}{a!} \frac{h^{b+1}}{(b+1)!} = \\
& \frac{y^{b+1}}{(b+1)!} \frac{z^a}{a!} + \sum_{k=1}^b (2a)^k \frac{y^{b+1-k}}{(b+1-k)!} \frac{z^a}{a!} \frac{h^k}{k!} + (2a)^{b+1} \frac{z^a}{a!} \frac{h^{b+1}}{(b+1)!} = \\
& \sum_{k=0}^{b+1} (2a)^k \frac{y^{b+1-k}}{(b+1-k)!} \frac{z^a}{a!} \frac{h^k}{k!}.
\end{aligned}$$

Next we show item 3. From Lemma 1 we have that

$$\left(y + (a + b - (v + 1))h \right)_{v, -h} = \sum_{w=0}^v (a + b - 2v)_v^w y^{v-w} h^w.$$

Thus our desired identity is equivalent to

$$\frac{z^a}{a!} \frac{x^b}{b!} = \sum_{v=0}^{\min(a,b)} (-1)^v \frac{x^{b-v}}{(b-v)!} \left(y + (a + b - (v + 1))h \right)_{v, -h} \frac{z^{a-v}}{(a-v)!} \frac{h^v}{v!}.$$

By induction on b one shows that:

$$z \frac{x^b}{b!} = \frac{x^b}{b!} z - \frac{x^{b-1}}{(b-1)!} (y + (b-1)h)h.$$

We proceed from this formula by induction on a .

$$\begin{aligned}
& \frac{z^{a+1}}{(a+1)!} \frac{x^b}{b!} = \frac{z}{(a+1)} \left(\frac{z^a}{a!} \frac{x^b}{b!} \right) = \\
& \sum_{v=0}^{\min(a,b)} \frac{(-1)^v}{(a+1)} \left(z \frac{x^{b-v}}{(b-v)!} \right) \left(y + (a + b - v - 1)h \right)_{v, -h} \frac{z^{a-v}}{(a-v)!} \frac{h^v}{v!} = \\
& \sum_{v=0}^{\min(a,b)} \frac{(-1)^v}{(a+1)} \left(\frac{x^{b-v}}{(b-v)!} z - \frac{x^{b-v-1}}{(b-v-1)!} (y + (b-v-1)h)h \right) \left(y + (a + b - v - 1)h \right)_{v, -h} \frac{z^{a-v}}{(a-v)!} \frac{h^v}{v!} = \\
& \sum_{v=0}^{\min(a,b)} \frac{(-1)^v}{(a+1)} \frac{x^{b-v}}{(b-v)!} z \left(y + (a + b - v - 1)h \right)_{v, -h} \frac{z^{a-v}}{(a-v)!} \frac{h^v}{v!} + \\
& \sum_{v=0}^{\min(a,b)} \frac{(-1)^{v+1}}{(a+1)} (v+1) \frac{x^{b-v-1}}{(b-v-1)!} \left(y + (b-v-1)h \right) \left(y + (a + b - v - 1)h \right)_{v, -h} \frac{z^{a-v}}{(a-v)!} \frac{h^{v+1}}{(v+1)!} = \\
& \sum_{v=0}^{\min(a,b)} \frac{(-1)^v}{(a+1)} \frac{x^{b-v}}{(b-v)!} \left(y + (a + b - v + 1)h \right)_{v, -h} z \frac{z^{a-v}}{(a-v)!} \frac{h^v}{v!} +
\end{aligned}$$

$$\begin{aligned}
& \sum_{v=0}^{\min(a,b)} \frac{(-1)^{v+1}}{(a+1)} (v+1) \frac{x^{b-v-1}}{(b-v-1)!} \left(y + (b-v-1)h \right) \left(y + (a+b-v-1)h \right)_{v,-h} \frac{z^{a-v}}{(a-v)!} \frac{h^{v+1}}{(v+1)!} = \\
& \sum_{v=0}^{\min(a,b)} \frac{(-1)^v}{(a+1)} (a+1-v) \frac{x^{b-v}}{(b-v)!} \left(y + (a+b-v+1)h \right)_{v,-h} \frac{z^{a+1-v}}{(a+1-v)!} \frac{h^v}{v!} + \\
& \sum_{v=0}^{\min(a,b)} \frac{(-1)^{v+1}}{(a+1)} (v+1) \frac{x^{b-v-1}}{(b-v-1)!} \left(y + (b-v-1)h \right) \left(y + (a+b-v-1)h \right)_{v,-h} \frac{z^{a-v}}{(a-v)!} \frac{h^{v+1}}{(v+1)!}.
\end{aligned}$$

Making the change $v \rightarrow v+1$ we get that $\frac{z^{a+1}}{(a+1)!} \frac{x^b}{b!}$ is equal to:

$$\begin{aligned}
& \sum_{v=0}^{\min(a,b)} \frac{(-1)^v}{(a+1)} (a+1-v) \frac{x^{b-v}}{(b-v)!} \left(y + (a+b-v+1)h \right)_{v,-h} \frac{z^{a+1-v}}{(a+1-v)!} \frac{h^v}{v!} + \\
& \sum_{v=1}^{\min(a+1,b)} \frac{(-1)^v}{(a+1)} v \frac{x^{b-v}}{(b-v)!} \left(y + (b-v)h \right) \left(y + (a+b-v)h \right)_{v-1,-h} \frac{z^{a+1-v}}{(a+1-v)!} \frac{h^v}{v!} = \\
& \frac{x^b}{b!} \frac{z^{a+1}}{(a+1)!} + \\
& \sum_{v=1}^{\min(a,b)} \frac{(-1)^v}{(a+1)} (a+1-v) \frac{x^{b-v}}{(b-v)!} \left(y + (a+b-v+1)h \right)_{v,-h} \frac{z^{a+1-v}}{(a+1-v)!} \frac{h^v}{v!} + \\
& \sum_{v=1}^{\min(a+1,b)} \frac{(-1)^v}{(a+1)} v \frac{x^{b-v}}{(b-v)!} \left(y + (b-v)h \right) \left(y + (a+b-v)h \right)_{v-1,-h} \frac{z^{a+1-v}}{(a+1-v)!} \frac{h^v}{v!} = \\
& \frac{x^b}{b!} \frac{z^{a+1}}{(a+1)!} + (-1)^{a+1} \frac{x^{b-a-1}}{(b-a-1)!} \left(y + (b-a-1)h \right) \left(y + (b-1)h \right)_{a,-h} \frac{h^{a+1}}{(a+1)!} + \\
& \sum_{v=1}^{\min(a,b)} \frac{(-1)^v}{(a+1)} (a+1-v) \frac{x^{b-v}}{(b-v)!} \left(y + (a+b-v+1)h \right)_{v,-h} \frac{z^{a+1-v}}{(a+1-v)!} \frac{h^v}{v!} + \\
& \sum_{v=1}^{\min(a,b)} \frac{(-1)^v}{(a+1)} v \frac{x^{b-v}}{(b-v)!} \left(y + (b-v)h \right) \left(y + (a+b-v)h \right)_{v-1,-h} \frac{z^{a+1-v}}{(a+1-v)!} \frac{h^v}{v!}.
\end{aligned}$$

We apply to the latter expression the following identities coming from Lemma 1

$$\begin{aligned}
& \left(y + (b-1)h \right)_{a+1,-h} = \left(y + (b-a-1)h \right) \left(y + (b-1)h \right)_{a,-h}, \\
& \left(y + (a+b-v+1)h \right)_{v,-h} = \left(y + (a+b-v+1)h \right) \left(y + (a+b-v+1)h - h \right)_{v-1,-h},
\end{aligned}$$

to get that

$$\frac{x^b}{b!} \frac{z^{a+1}}{(a+1)!} + (-1)^{a+1} \frac{x^{b-a-1}}{(b-a-1)!} \left(y + (b-1)h \right)_{a+1,-h} \frac{h^{a+1}}{(a+1)!} +$$

$$\sum_{v=1}^{\min(a,b)} \frac{(-1)^v}{(a+1)} \frac{x^{b-v}}{(b-v)!} (a+1-v) \left(y + (a+b-v+1)h \right) \left(y + (a+b-v)h \right)_{v-1,-h} \frac{z^{a+1-v}}{(a+1-v)!} \frac{h^v}{v!} +$$

$$\sum_{v=1}^{\min(a,b)} \frac{(-1)^v}{(a+1)} \frac{x^{b-v}}{(b-v)!} \left(v(y + (b-v)h) (y + (a+b-v)h)_{v-1,-h} \right) \frac{z^{a+1-v}}{(a+1-v)!} \frac{h^v}{v!}.$$

Taking $\frac{x^{b-v}}{(b-v)!}$, $\frac{z^{a+1-v}}{(a+1-v)!}$, $\frac{h^v}{v!}$ and $\left(y + (a+b-v)h \right)_{v-1,-h}$ as a common factors, the previous expression becomes:

$$\frac{x^b}{b!} \frac{z^{a+1}}{(a+1)!} + (-1)^{a+1} \frac{x^{b-a-1}}{(b-a-1)!} \left(y + (b-1)h \right)_{a+1,-h} \frac{h^{a+1}}{(a+1)!} +$$

$$\sum_{v=1}^{\min(a,b)} \frac{(-1)^v}{(a+1)} \frac{x^{b-v}}{(b-v)!} (a+1) \left(y + (a+b-2v+1)h \right) \left(y + (a+b-v)h \right)_{v-1,-h} \frac{z^{a+1-v}}{(a+1-v)!} \frac{h^v}{v!} =$$

$$\frac{x^b}{b!} \frac{z^{a+1}}{(a+1)!} + (-1)^{a+1} \frac{x^{b-a-1}}{(b-a-1)!} \left(y + (b-1)h \right)_{a+1,-h} \frac{h^{a+1}}{(a+1)!} +$$

$$\sum_{v=1}^{\min(a,b)} (-1)^v \frac{x^{b-v}}{(b-v)!} \left(y + (a+b-2v+1)h \right) \left(y + (a+b-v)h \right)_{v-1,-h} \frac{z^{a+1-v}}{(a+1-v)!} \frac{h^v}{v!}.$$

We apply the following identity coming from Lemma 1

$$\left(y + (a+b-v)h \right)_{v,-h} = \left(y + (a+b-v)h \right)_{v-1,-h} \left(y + (a+b-2v-1)h \right)$$

to the third term above to get that:

$$\frac{x^b}{b!} \frac{z^{a+1}}{(a+1)!} + (-1)^{a+1} \frac{x^{b-a-1}}{(b-a-1)!} \left(y + (b-1)h \right)_{a+1,-h} \frac{h^{a+1}}{(a+1)!} +$$

$$\sum_{v=1}^{\min(a,b)} (-1)^v \frac{x^{b-v}}{(b-v)!} \left(y + (a+b-v)h \right)_{v,-h} \frac{z^{a+1-v}}{(a+1-v)!} \frac{h^v}{v!} =$$

$$\sum_{v=0}^{\min(a+1,b)} (-1)^v \frac{x^{b-v}}{(b-v)!} \left(y + (a+b-v)h \right)_{v,-h} \frac{z^{a+1-v}}{(a+1-v)!} \frac{h^v}{v!},$$

showing the desired result. □

Example 3. For $a = 1$ and $b = 2$, the identities defining $\widehat{U}_h(\mathfrak{sl}_2)$ imply that:

$$z \frac{x^2}{2!} = (zx) \frac{x}{2!} = (xz - yh) \frac{x}{2!} = \frac{x}{2!} (zx) - (yx) \frac{h}{2!} =$$

$$\frac{x}{2!} (xz - yh) - (xy + 2xh) \frac{h}{2!} = \frac{x^2}{2!} z - \frac{1}{2!} xyh - \frac{1}{2!} xyh - 2x \frac{h^2}{2!} =$$

$$\frac{x^2}{2!}z - xyh - 2x\frac{h^2}{2!}.$$

On the other hand, using Proposition 2 we get that:

$$\begin{aligned} z\frac{x^2}{2!} &= \sum_{0 \leq w \leq v \leq 1} (-1)^v (v-w)! (v+w)_w (3-2v)_v \frac{x^{2-v}}{(2-v)!} \frac{y^{v-w}}{(v-w)!} \frac{z^{1-v}}{(1-v)!} \frac{h^{v+w}}{(v+w)!} = \\ &= (-1)^0 (0)!(0)_0 (3)_0^0 \frac{x^2}{2!} \frac{y^0}{0!} \frac{z^1}{1!} \frac{h^0}{0!} + (-1)^1 (1)!(1)_0 (1)_1^0 \frac{x^1}{1!} \frac{y^1}{1!} \frac{z^0}{0!} \frac{h^1}{1!} + (-1)^1 (0)!(2)_1 (1)_1^1 \frac{x^1}{1!} \frac{y^0}{0!} \frac{z^0}{0!} \frac{h^2}{2!} = \\ &= \frac{x^2}{2!}z - xyh - 2x\frac{h^2}{2!}. \end{aligned}$$

We are ready to show the main result of this work, namely, an explicit formula for the product on $\widehat{U}_h(\mathfrak{sl}_2)$ in the divided power basis. The formula itself may look unwieldy at first, but we show in Section 3 that it has a transparent combinatorial meaning. We denote it by \star the product on $\widehat{U}_h(\mathfrak{sl}_2)$ in order to distinguish it from the commutative product of formal power series.

Theorem 4. Let $f, g \in \widehat{U}_h(\mathfrak{sl}_2)$ be given by

$$f = \sum_{a,b,c,d \in \mathbb{N}} f_{a,b,c,d} \frac{x^a y^b z^c h^d}{a!b!c!d!} \quad \text{and} \quad g = \sum_{k,l,m,n \in \mathbb{N}} g_{k,l,m,n} \frac{x^k y^l z^m h^n}{k!l!m!n!}.$$

The product $f \star g \in \widehat{U}_h(\mathfrak{sl}_2)$ is given by

$$f \star g = \sum_{\alpha, \beta, \gamma, \rho \in \mathbb{N}} (f \star g)_{\alpha, \beta, \gamma, \rho} \frac{x^\alpha y^\beta z^\gamma h^\rho}{\alpha! \beta! \gamma! \rho!}$$

where the coefficients $(f \star g)_{\alpha, \beta, \gamma, \rho}$ are defined using 13 auxiliary variables

$$\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6 \in \mathbb{N}^{13} \quad \text{such that:}$$

$$\alpha_1 + \alpha_2 = \alpha, \quad \beta_1 + \beta_2 + \beta_3 = \beta, \quad \gamma_1 + \gamma_2 = \gamma,$$

$$\rho_1 + \rho_2 + \rho_3 + \rho_4 + \rho_5 + \rho_6 = \rho, \quad \beta_3 + \rho_4 = \rho_3.$$

The constant $(f \star g)_{\alpha, \beta, \gamma, \rho}$ is given by

$$\begin{aligned} &\sum f_{\alpha_1, \beta_1 + \rho_5, \gamma_1 + \rho_3, \rho_1} g_{\alpha_2 + \rho_3, \beta_3 + \rho_6, \gamma_2, \rho_2} \begin{pmatrix} \alpha \\ \alpha_1, \alpha_2 \end{pmatrix} \begin{pmatrix} \beta \\ \beta_1, \beta_2, \beta_3 \end{pmatrix} \begin{pmatrix} \gamma \\ \gamma_1, \gamma_2 \end{pmatrix} \\ &(-1)^{\rho_3} \begin{pmatrix} \rho \\ \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6 \end{pmatrix} (2\alpha_2)^{\rho_5} (2\gamma_1)^{\rho_6} \beta_2! \rho_4! (\gamma_1 + \alpha_2)_{\beta_3 + \rho_4}^{\rho_4}, \end{aligned}$$

where the broken line means multiplication.

Proof. The result follows after several applications of Proposition 2. By definition the product $f \star g$ is equal to

$$\sum_{a,b,c,d,k,l,m,n} f_{a,b,c,d} g_{k,l,m,n} \left(\frac{x^a}{a!} \frac{y^b}{b!} \frac{z^c}{c!} \frac{h^d}{d!} \right) \star \left(\frac{x^k}{k!} \frac{y^l}{l!} \frac{z^m}{m!} \frac{h^n}{n!} \right).$$

Since h commutes with the other variables, we put together the h -monomials and get

$$\sum \left[f_{a,b,c,d} g_{k,l,m,n} \binom{d+n}{d,n} \right] \frac{x^a}{a!} \frac{y^b}{b!} \left(\frac{z^c}{c!} \frac{x^k}{k!} \right) \frac{y^l}{l!} \frac{z^m}{m!} \frac{h^{d+n}}{(d+n)!}.$$

Using Proposition 2 we order the selected z and x monomials obtaining

$$\sum \left[f_{a,b,c,d} g_{k,l,m,n} \binom{d+n}{d,n} (-1)^v (v-w)! (v+w)_w (c+k-2v)_v^w \right] \frac{x^a}{a!} \frac{y^b}{b!} \left(\frac{x^{k-v}}{(k-v)!} \frac{y^{v-w}}{(v-w)!} \frac{z^{c-v}}{(c-v)!} \frac{h^{v+w}}{(v+w)!} \right) \frac{y^l}{l!} \frac{z^m}{m!} \frac{h^{d+n}}{(d+n)!}$$

where $0 \leq w \leq v \leq \min(c, k)$. We collect h -monomials together and get:

$$\sum \left[f_{a,b,c,d} g_{k,l,m,n} \binom{d+n}{d,n} \binom{d+n+v+w}{d+n, v+w} (-1)^v (v-w)! (v+w)_w (c+k-2v)_v^w \right] \frac{x^a}{a!} \left(\frac{y^b}{b!} \frac{x^{k-v}}{(k-v)!} \right) \frac{y^{v-w}}{(v-w)!} \frac{z^{c-v}}{(c-v)!} \frac{y^l}{l!} \frac{z^m}{m!} \frac{h^{d+n+v+w}}{(d+n+v+w)!}.$$

Using Proposition 2 we write the selected y and x monomials in normal order

$$\sum \left[f_{a,b,c,d} g_{k,l,m,n} \binom{d+n+v+w}{d,n, v+w} (-1)^v (v-w)! (v+w)_w (c+k-2v)_v^w (2(k-v))^i \right] \frac{x^a}{a!} \left(\frac{x^{k-v}}{(k-v)!} \frac{y^{b-i}}{(b-i)!} \frac{h^i}{i!} \right) \frac{y^{v-w}}{(v-w)!} \frac{z^{c-v}}{(c-v)!} \frac{y^l}{l!} \frac{z^m}{m!} \frac{h^{d+n+v+w}}{(d+n+v+w)!}$$

with $0 \leq i \leq b$. Collecting h -monomials we get:

$$\sum f_{a,b,c,d} g_{k,l,m,n} \binom{d+n+v+w+i}{d,n, v+w, i} \binom{a+k-v}{a, k-v} \binom{b-i+v-w}{b-i, v-w} (-1)^v (v-w)! (v+w)_w (c+k-2v)_v^w (2(k-v))^i \frac{x^{a+k-v}}{(a+k-v)!} \frac{y^{b-i+v-w}}{(b-i+v-w)!} \left(\frac{z^{c-v}}{(c-v)!} \frac{y^l}{l!} \right) \frac{z^m}{m!} \frac{h^{d+n+v+w+i}}{(d+n+v+w+i)!}.$$

Using Proposition 2 to write the selected z and y monomials in normal order, and collecting h -monomials we get:

$$\sum f_{a,b,c,d} g_{k,l,m,n} \binom{d+n+v+w+i+u}{d, n, v+w, i, u} \binom{c-v+m}{c-v, m} \binom{b-i+v-w+l-u}{b-i, v-w, l-u} \\ (-1)^v \binom{a+k-v}{a, k-v} (v-w)! (v+w)_w (c+k-2v)_v^w (2(k-v))^i (2(c-v))^u \\ \frac{x^{a+k-v}}{(a+k-v)!} \frac{y^{b-i+v-w+l-u}}{(b-i+v-w+l-u)!} \frac{z^{c-v+m}}{(c-v+m)!} \frac{h^{d+n+v+w+i+u}}{(d+n+v+w+i+u)!}$$

with $0 \leq u \leq l$. Using $(v+w)_w = \binom{v+w}{w} w!$, we obtain:

$$\sum f_{a,b,c,d} g_{k,l,m,n} \binom{d+n+v+w+i+u}{d, n, v, w, i, u} \binom{c-v+m}{c-v, m} \binom{b-i+v-w+l-u}{b-i, v-w, l-u} \\ (-1)^v \binom{a+k-v}{a, k-v} (v-w)! w! (c+k-2v)_v^w (2(k-v))^i (2(c-v))^u \\ \frac{x^{a+k-v}}{(a+k-v)!} \frac{y^{b-i+v-w+l-u}}{(b-i+v-w+l-u)!} \frac{z^{c-v+m}}{(c-v+m)!} \frac{h^{d+n+v+w+i+u}}{(d+n+v+w+i+u)!}.$$

Finally performing the change of variables specified below we obtain that:

$$\sum f_{\alpha_1, \beta_1 + \rho_5, \gamma_1 + \rho_3, \rho_1} g_{\alpha_2 + \rho_3, \beta_3 + \rho_6, \gamma_2, \rho_2} \binom{\rho}{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6} \binom{\gamma}{\gamma_1, \gamma_2} \binom{\beta}{\beta_1, \beta_2, \beta_3} \binom{\alpha}{\alpha_1, \alpha_2} \\ \left[(-1)^{\rho_3} \beta_2! \rho_4! (\gamma_1 + \alpha_2)_{\rho_3}^{\rho_4} (2\alpha_2)^{\rho_5} (2\gamma_1)^{\rho_6} \right] \frac{x^\alpha y^\beta z^\gamma h^\rho}{\alpha! \beta! \gamma! \rho!},$$

where

$$\alpha_1 = a, \quad \alpha_2 = k - v, \quad \beta_1 = b - i, \quad \beta_2 = l - u, \quad \beta_3 = v - w, \quad \gamma_1 = c - v, \quad \gamma_2 = m, \\ \rho_1 = d, \quad \rho_2 = n, \quad \rho_3 = v, \quad \rho_4 = w, \quad \rho_5 = i, \quad \rho_6 = u, \quad \beta_3 + \rho_4 = \rho_3.$$

□

3 Combinatorial approach towards $\widehat{U}_h(\mathfrak{sl}_2)$

The formula for the product on $\widehat{U}_h(\mathfrak{sl}_2)$ given in Theorem 4 although explicit is quite hard to digest. The presence of 13 auxiliary variables may suggest that an intuitive understanding of this product is simply hopeless. In this section we argue on the contrary, and show that a sophisticated but easy to grasp combinatorial understanding of the product rule is indeed possible.

We are going to phrase our results in the language of the theory of species introduced by Joyal [16, 17]. The reader may consult [2] for a comprehensive introduction to the subject, including a fairly exhaustive list of references. Here we just briefly introduce the main ideas needed in order to make this work reasonably self-contained, following the categorical approach developed in [4, 5, 8].

Let \mathbb{B} be the category of finite sets and bijections, i.e. \mathbb{B} is the underlying groupoid of the category \mathbf{set} of finite sets and maps. For $d \geq 1$, let \mathbb{B}^d be the d -fold Cartesian product of \mathbb{B} with itself. Objects of \mathbb{B}^d are d -tuples of finite sets, and may also be regarded as pairs (a, f) , where a is a finite set and $f : a \rightarrow [d]$ is a map. A morphism from (a, f) to (b, g) is a bijection $\alpha : a \rightarrow b$ such that $g\alpha = f$. We think of $[d]$ as a set of d colors, and (a, f) as a colored set.

Let \mathcal{C} be a distributive category, meaning that \mathcal{C} comes with functors \oplus and \otimes satisfying suitable axioms. Coherence laws for such structures have been introduced by Laplaza [20, 21]. For the purposes of this work the reader may take \mathcal{C} to be the category of finite sets and maps, or a category of finite dimensional vector spaces and linear transformations over a field. In the latter case \oplus and \otimes are, respectively, direct sum and tensor product of vector spaces, whereas in the former case \oplus and \otimes are disjoint union and Cartesian product, and thus are denoted by \sqcup and \times .

We also demand that a negative functor $- : \mathcal{C} \rightarrow \mathcal{C}$ be defined on \mathcal{C} , which is assumed to come with natural isomorphisms:

$$\begin{aligned} -0 &\simeq 0, & -(a \oplus b) &\simeq -a \oplus -b, \\ (-a) \otimes b &\simeq -(a \otimes b) \simeq a \otimes (-b) && \text{for } a, b \in \mathcal{C}. \end{aligned}$$

There is a simple mechanism, akin to Grothendick's construction of the group associated to an abelian monoid, enhancing a distributive category into a distributive category with a negative functor. Namely, given \mathcal{C} one considers the Cartesian product category $\mathbb{Z}_2\text{-}\mathcal{C} = \mathcal{C} \times \mathcal{C}$. The sum, negative, and product functors on $\mathbb{Z}_2\text{-}\mathcal{C}$ are given on objects $a_1, a_2, b_1, b_2 \in \mathcal{C}$ by

$$(a_1, a_2) \oplus (b_1, b_2) = (a_1 \oplus b_1, a_2 \oplus b_2), \quad -(a_1, a_2) = (a_2, a_1),$$

$$(a_1, a_2) \otimes (b_1, b_2) = (a_1 \otimes b_1 \oplus a_2 \otimes b_2, a_1 \otimes b_2 \oplus a_2 \otimes b_1),$$

and naturally extended to morphisms. There is an inclusion functor $i : C \rightarrow \mathbb{Z}_2\text{-}C$ given on objects and morphisms by $i(a) = (a, 0)$ and $i(f) = (f, 1_a)$.

Note that we have a functor $\text{set} \rightarrow C$ which sends a finite set x to the object $\bigoplus_{i \in x} k$ of C . This functor allow us to transport combinatorial constructions to C .

Consider the category $[\mathbb{B}^d, C]$ of functors from \mathbb{B}^d to C , and natural transformations as morphisms. There are several interesting structural functors on $[\mathbb{B}^d, C]$ some of which we proceed to briefly describe. These structural functors on C are inspired by the close relationship between functors in $[\mathbb{B}^d, C]$ and formal power series in d -variables defined via generating series.

We assume that C comes with a valuation map $|\cdot| : C \rightarrow R$ where R is a ring with $R \supseteq \mathbb{Q}$. For example, for the category of finite sets we let $|x|$ be the cardinality of x , and for the category of finite dimensional vector spaces we let $|V|$ be the dimension of V . The valuation map should satisfy the following identities for $a, b \in C$:

$$|a| = |b| \quad \text{if} \quad a \simeq b, \quad |a \oplus b| = |a| + |b|,$$

$$|a \otimes b| = |a||b|, \quad |1| = 1, \quad |0| = 0, \quad \text{and} \quad |-a| = -|a|.$$

The valuation map on C can be extended to a map

$$|\cdot| : [\mathbb{B}^d, C] \rightarrow R[[x_1, \dots, x_d]] \quad \text{which sends a functor to its generating series}$$

$$|F| = \sum_{(n_1, \dots, n_d) \in \mathbb{N}^d} |F([n_1], \dots, [n_d])| \frac{x_1^{n_1} \dots x_d^{n_d}}{n_1! \dots n_d!}.$$

Next we introduce further structures present on the category $[\mathbb{B}^d, C]$.

- The **sum functor** $F + G \in [\mathbb{B}^d, C]$ is given on $(x, f) \in \mathbb{B}^d$ by

$$(F + G)(x, f) = F(x, f) + G(x, f),$$

and is such that $|F + G| = |F| + |G|$.

- The **negative functor** $-F \in [\mathbb{B}^d, C]$ is given on $(x, f) \in \mathbb{B}^d$ by

$$(-F)(x, f) = -(F(x, f)),$$

and is such that $|-F| = -|F|$.

- The **Hadamard product** functor $F \times G \in [\mathbb{B}^d, C]$ is given on $(x, f) \in \mathbb{B}^d$ by

$$F \times G(x, f) = F(x, f) \otimes G(x, f),$$

and is such that $|F \times G| = |F| \times |G|$ where the Hadamard product on series is given by coefficient-wise multiplication.

- The **product** functor $FG \in [\mathbb{B}^d, C]$ is given on $(x, f) \in \mathbb{B}^d$ by

$$FG(x, f) = \bigoplus_{(x_1, f_1) \sqcup (x_2, f_2) = (x, f)} F(x_1, f_1) \otimes G(x_2, f_2),$$

and is such that $|FG| = |F||G|$.

- The **composition** or **substitution** functor $F(G_1, \dots, G_d) \in [\mathbb{B}^d, C]$ is given on $(x, f) \in \mathbb{B}^d$ by

$$F(G_1, \dots, G_d)(x, f) = \bigoplus_{\substack{\pi \in \text{Par}(x) \\ c: \pi \rightarrow [d]}} F(\pi, c) \otimes \bigotimes_{a \in \pi} G_{c(a)}(a, f|_a),$$

where we assume that $G_i(\emptyset) = 0$. We have that $|F(G_1, \dots, G_d)| = |F|(|G_1|, \dots, |G_d|)$.

- A **quantum \star -product** on a suitable category of functors was introduced in [8] in order to produce a categorification of the formal homogeneous Weyl algebras. Explicitly, for $d \geq 1$, consider the category of functors

$$[\mathbb{B}^{2d+1}, C] = [\mathbb{B}^d \times \mathbb{B}^d \times \mathbb{B}, C].$$

We regard objects of $\mathbb{B}^d \times \mathbb{B}^d \times \mathbb{B}$ as triples (x, f, h) where x and h are finite sets, and $f: x \rightarrow [d] \sqcup [\tilde{d}]$ is a map. The \star -product $F \star G \in [\mathbb{B}^{2d+1}, C]$ of functors $F, G \in [\mathbb{B}^{2d+1}, C]$ is given by

$$F \star G(x, f, h) = \bigoplus F(x_1 \sqcup h_3, f|_{x_1} \sqcup \tilde{g}, h_1) \otimes G(x_2 \sqcup h_3, f|_{x_2} \sqcup g, h_2)$$

where the sum runs over all pairs x_1, x_2 and all triples h_1, h_2, h_3 such that

$$x_1 \sqcup x_2 = x, \quad h_1 \sqcup h_2 \sqcup h_3 = h \quad \text{and} \quad g: h_3 \rightarrow [d].$$

Where $f|_{x_1}$ and $f|_{x_2}$ are the restriction maps, and \tilde{g} has the same domain as g , and assumes the corresponding values in $[\tilde{d}] = \{\tilde{1}, \dots, \tilde{d}\}$. This product is a categorification of the formal homogeneous Weyl algebra in the sense that we have a map

$$| \cdot | : [\mathbb{B}^{2d+1}, C] \rightarrow \mathbb{W}_n(R)$$

sending a functor to its generating function $|F|$ given by

$$\sum_{(n_1, \dots, n_d, m_1, \dots, m_d, k) \in \mathbb{N}^{2d+1}} |F([n_1], \dots, [n_d], [m_1], \dots, [m_d], [k])| \frac{x_1^{n_1} \dots x_d^{n_d} y_1^{m_1} \dots y_d^{m_d} h^k}{n_1! \dots n_d! m_1! \dots m_d! k!},$$

which satisfies that

$$|F \star G| = |F| \star |G|.$$

On the right hand side of this equation the \star -product of formal power series is the product of formal power series in the formal homogeneous Weyl algebra:

$$\mathbb{W}_n(R) = R \langle\langle x_1, \dots, x_d, y_1, \dots, y_d, h \rangle\rangle / I_d, \quad \text{where:}$$

- $R \langle\langle x_1, \dots, x_d, y_1, \dots, y_d, h \rangle\rangle$ is the ring of formal power series with coefficients in R in the non-commutative variables $x_1, \dots, x_d, y_1, \dots, y_d, h$.
- I_d is the ideal generated by the relations:
 - * h commutes with all other variables.
 - * $x_j x_i = x_i x_j$ and $y_j y_i = y_i y_j$, for $i, j \in [d]$.
 - * $y_j x_i = x_i y_j$, for $i \neq j \in [d]$.
 - * $y_i x_i = x_i y_i + h$, for $i \in [d]$.

Next we introduce a few examples of functors that will be needed in our construction of the combinatorial counterpart of the product on $\widehat{U}_h(\mathfrak{sl}_2)$.

- The functor of linear orderings $\mathbb{L} : \mathbb{B} \rightarrow \text{set}$ is given on objects by

$$\mathbb{L}(x) = \{ \alpha : [x] \rightarrow x \mid \alpha \text{ bijective} \}.$$

The generating series of \mathbb{L} is $\frac{1}{1-x}$.

- The functor of maps $[\ , \] : \mathbb{B}^2 \rightarrow \text{set}$ is given on objects by

$$[x, y] = \{ f \mid f : x \rightarrow y \text{ is a map} \}.$$

The generating series of $[\ , \]$ is given by $|[\ , \]| = e^{ye^x}$.

- The functor $(\bullet)^\bullet_\bullet : \mathbb{B}^3 \rightarrow \text{set}$ is such that $(x)_y^z$ is the subset of

$$[[z], \bigsqcup_{i=0}^{|y|-1} x \sqcup [i]]$$

consisting of those maps such that:

- For $s < t$ in $[z]$, if s is mapped into the component $x \sqcup [i]$, then t is mapped into a component $x \sqcup [j]$ with $i < j$.
- By convention $(x)_\emptyset^\emptyset$ is a one element set, and $(x)_\emptyset^z = \emptyset$ for $z \neq \emptyset$.

The generating function of $(\bullet)^\bullet_\bullet$ is given by

$$|(\bullet)^\bullet_\bullet| = \sum_{n,m,k \in \mathbb{N}} (n)_m^k \frac{x^n y^m z^k}{n! m! k!}.$$

- Below we use the functor $\mathbb{B}^4 \rightarrow C$ sending a tuple of finite sets (a, b, c, d) to $(a \sqcup b)_{c \sqcup d}^d$.

- The x -colored singleton functor $X : \mathbb{B}^4 \rightarrow C$ evaluated on (x, y, z, h) is 1 if $(|x|, |y|, |z|, |h|) = (1, 0, 0, 0)$, and 0 otherwise. Singleton functors Y , Z , and H are defined analogously. The corresponding generating series are given by $|X| = x$, $|Y| = y$, $|Z| = z$, $|H| = h$.
- The divided power functor $\frac{X^a}{a!} : \mathbb{B}^4 \rightarrow \mathbb{C}$, for $a \geq 0$, is given by

$$\frac{X^a}{a!}(x, y, z, h) = \begin{cases} 1 & \text{if } |x| = a, y = z = h = \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

with analogue definitions for the divided powers functors $\frac{Y^a}{a!}$, $\frac{Z^a}{a!}$, $\frac{H^a}{a!}$. The corresponding generating series are given by $|\frac{X^a}{a!}| = \frac{x^a}{a!}$, $|\frac{Y^a}{a!}| = \frac{y^a}{a!}$, $|\frac{Z^a}{a!}| = \frac{z^a}{a!}$, $|\frac{H^a}{a!}| = \frac{h^a}{a!}$.

Recall that we are fixing a distributive category with negative objects C which comes with a valuation taking values in a ring $R \supseteq \mathbb{Q}$. Our next goal is to introduce a \star -product on the category $[\mathbb{B}^4, C]$ which encodes the combinatorial properties of the product on $\widehat{U}_h(\mathfrak{sl}_2)$. We think of an object in \mathbb{B}^4 either as a 4-tuple of finite sets, or as a 4-colored finite set with the following conventions: red represents the variable x , purple represents the variable y , green represents the variable z , and blue represents the variable h .

Definition 5. Let F, G be functors $\mathbb{B}^4 \rightarrow C$, the \star -product functor $F \star G : \mathbb{B}^4 \rightarrow C$ sends a tuple $(x, y, z, h) \in \mathbb{B}^4$ to the object $F \star G(x, y, z, h) \in C$ given by

$$\bigoplus (-1)^{|h_3|} F(x_1, y_1 \sqcup h_5, z_1 \sqcup h_3, h_1) \otimes G(x_2 \sqcup h_3, y_2 \sqcup h_6, z_2, h_2) \otimes \mathbb{M}(x_2, z_1, y_3, h_4, h_5, h_6)$$

where the sum runs over the tuples

$$x_1, x_2, y_1, y_2, y_3, z_1, z_2, h_1, h_2, h_3, h_4, h_5, h_6 \in \mathbb{B}^{13}$$

such that

$$\begin{aligned} x_1 \sqcup x_2 &= x, & y_1 \sqcup y_2 \sqcup y_3 &= y, & z_1 \sqcup z_2 &= z, \\ h_1 \sqcup h_2 \sqcup h_3 \sqcup h_4 \sqcup h_5 \sqcup h_6 &= h, & |y_3| + |h_4| &= h_3, \end{aligned}$$

and the functor $\mathbb{M} : \mathbb{B}^6 \rightarrow C$ is given on objects by

$$\mathbb{M}(x_2, z_1, y_3, h_4, h_5, h_6) = [h_5, x_2 \sqcup x_2] \otimes [h_2, z_1 \sqcup z_1] \otimes \mathbb{L}(y_3) \otimes \mathbb{L}(h_4) \otimes (z_1 \sqcup x_2)_{y_3 \sqcup h_4}^{y_3}.$$

Proposition 6. The product $F \star G$ is indeed a functor $\mathbb{B}^4 \rightarrow C$.

Proof. We must show that a 4-tuple of bijections from (x, y, z, h) to (a, b, c, d) induces a map $F \star G(x, y, z, h) \rightarrow F \star G(a, b, c, d)$, and that this correspondence respects composition. The result follows since the structures involved in the construction of $F \star G$ (partitions of sets, maps between sets, linear orderings, F -structures, G -structures, and $(\bullet)_\bullet$ -structures) are transportable under bijections. \square

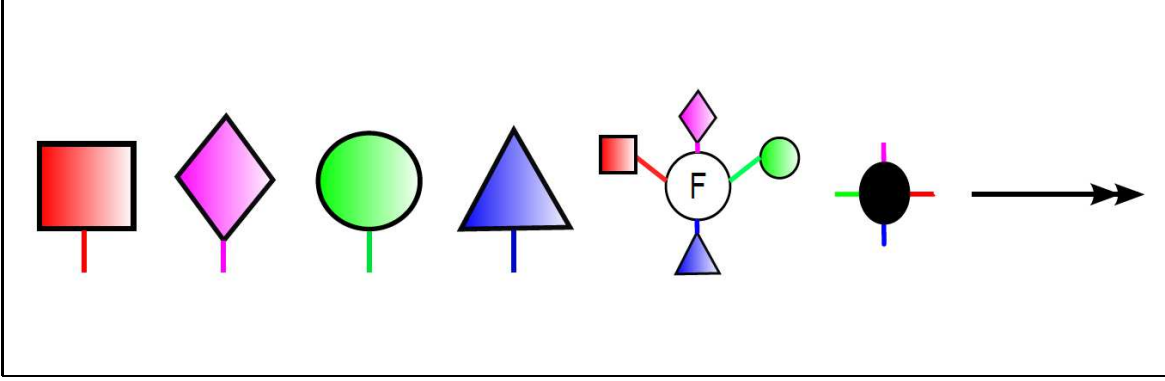


Figure 2: Basic ingredients for the graphical interpretation.

The category $[\mathbb{B}^4, C]$ comes equipped with a natural map

$$| \cdot | : [\mathbb{B}^4, C] \longrightarrow \widehat{U}_h(\mathfrak{sl}_2) \quad \text{given by}$$

$$|F| = \sum_{a,b,c,d \in \mathbb{N}} |F([a], [b], [c], [d])| \frac{x^a y^b z^c h^d}{a! b! c! d!}.$$

Theorem 7. For $F, G \in [\mathbb{B}^4, C]$ we have that $|F \star G| = |F| \star |G|$.

Proof. The result follows by correlating the various ingredients taking part in Definition 5 and Theorem 4:

- The partitions in Definition 5 give rise to the sum, the binomial coefficients, and the multinomial coefficients in Theorem 4. The same sign $(-)^{|h_3|} = (-1)^{\rho_3}$ is applied in both cases.
- The \otimes -factors $F(x_1, y_1 \sqcup h_5, z_1 \sqcup h_3, h_1) \otimes G(x_2 \sqcup h_3, y_2 \sqcup h_6, z_2, h_2)$ in Definition 5 give rise to the factors $f_{\alpha_1, \beta_1 + \rho_5, \gamma_1 + \rho_3, \rho_1} g_{\alpha_2 + \rho_3, \beta_3 + \rho_6, \gamma_2, \rho_2}$ in Theorem 4.
- The \otimes -factors $[h_5, x_2 \sqcup x_2] \otimes [h_6, z_1 \sqcup z_1]$ in Definition 5 give rise to the factors $(2\alpha_2)^{\rho_5} (2\gamma_1)^{\rho_6}$ in Theorem 4.
- The \otimes -factors $\mathbb{L}(y_3) \otimes \mathbb{L}(h_4)$ in Definition 5 give rise to the factorial factors $\beta_2! \rho_4!$ in Theorem 4.
- The \otimes -factor $(z_1 \sqcup x_2)_{y_3 \sqcup h_4}^{y_3}$ in Definition 5 gives rise to factor $(\gamma_1 + \alpha_2)_{\beta_3 + \rho_4}^{\rho_4}$ in Theorem 4.

□

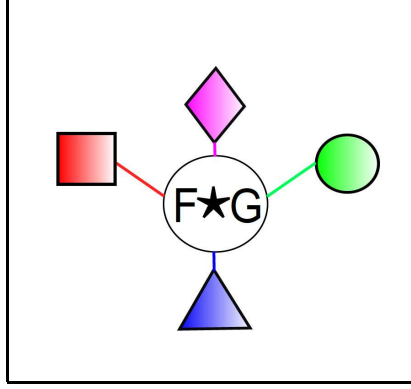


Figure 3: Diagrammatic representation of an application of the \star -product $F \star G$.

4 Graphs and the \star -product on $[\mathbb{B}^4, C]$

In this final section we introduce a graphical interpretation for the \star -product on $[\mathbb{B}^4, C]$, and thus we obtain, via Theorem 7, a combinatorial interpretation for the product on $\widehat{U}_h(\mathfrak{sl}_2)$. Let us introduce the basic ingredients, shown in Figure 2, from which we construct the kind of graphs that we are going to need.

- The red square with a line attached to it represents a x -colored set. If one wishes to be more specific we draw as many red lines as elements are in the set. Generically we draw only one line which stands for a multiplicity of lines. The same remark applies to the other basic components of our graphical constructions.
- The purple diamond with a line attached to it represents a y -colored set.
- The green disk with a line attached to it represents a z -colored set.
- The blue triangle with a line attached to it represents a h -colored set.
- The blob marked with the functor $F : \mathbb{B}^4 \rightarrow C$ represents the application of F to the disjoint union of the colored sets attached to it. Note that the color, the kind of figures, and even the position of attachment indicates the kind of variable represented by the various elements attached to the blob F . Figure 3 shows the diagram representing an application of the functor $F \star G$.
- The black disk represents the application of the functor $\mathbb{L}(y_3) \otimes \mathbb{L}(h_4) \otimes (z_1 \sqcup x_2)_{y_3 \sqcup h_4}^{y_3}$ to the sets attached to it, considered as an ordered tuple of sets using the counter clockwise cyclic order and starting from the set attached at the West position.
- A double arrow line represents the applications of the functor $\mathbb{B}^2 \rightarrow C$ sending (a, b) to $[a, b \sqcup b]$, where a and b are the incoming and outgoing sets linked by the arrow.

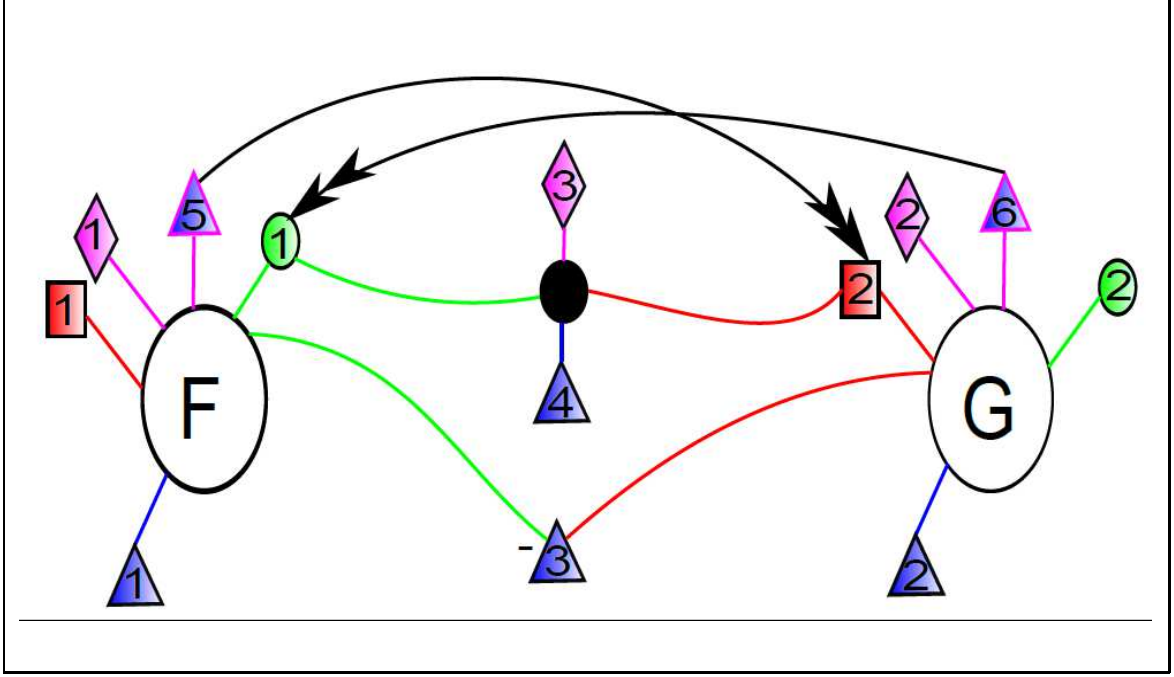


Figure 4: Graphical representation of the \star -product functor $F \star G$.

Proposition 8. The product $F \star G$ is represented, explicitly, by the graph in Figure 4.

Proof. We have to check that the various components of the graph from Figure 4 are in correspondence with the \otimes -terms of the product $F \star G$ as given in Definition 5. We proceed to analyze the various components of our graph.

- The red squares numbered 1 and 2 represent the partition of the x -colored set in two blocks.
- The purple diamonds numbered 1, 2 and 3 represent the partition of the y -colored set in three blocks.
- The green disks numbered 1 and 2 represent the partition of the z -colored set in two blocks.
- The blue triangles numbered 1 through 5 represent the partition of the h -colored set in five blocks.
- The blob marked by F represents the application of the functor F to the sets attached to it, which are x_1 , $y_1 \sqcup h_5$ (where the block h_5 changes from a blue z -color to a purple y -color), $z_1 \sqcup h_3$, and h_1 . Note that the block h_3 becomes a set of bi-colored edges starting as green y -edges and ending up as a red x -edges.

- The blob marked G represents the application of the functor G to the sets attached to it, namely $h_3 \sqcup x_2$, $y_2 \sqcup h_2$ (again the block h_2 changes from a blue z -color to a purple y -color), z_2 , and h_2 .
- The black disk with the various edges attached to it represents the application of the functor $\mathbb{L}(y_3) \otimes \mathbb{L}(h_4) \otimes (z_1 \sqcup x_2)_{y_3 \sqcup h_4}^{y_3}$.
- The double pointed arrows represent $[h_5, x_2 \sqcup x_2]$ and $[h_2, z_1 \sqcup z_1]$, respectively.
- The negative sign comes from the cardinality of the block h_3 .
- The condition $|y_3| + |h_4| = |h_3|$, implies that if h_3 is empty, then y_3 and h_4 are also empty; and that if the block h_3 is not empty, then y_3 and h_4 can not be both empty.

□

Next we put the graphical notation in action, thereby showing that it is an effective computational tool. Recall that the colored singular functors X, Y, Z, H output 0 unless applied to a set of cardinality 1 of the respective color where it outputs 1.

4.1 Graphs and the defining identities of $\widehat{U}_h(\mathfrak{sl}_2)$

Let us study the graphical interpretation of the defining identities for $\widehat{U}_h(\mathfrak{sl}_2)$.

Proposition 9. Consider the singular functors $X, Y, Z, H \in [\mathbb{B}^4, C]$. We have that

1. The functor $Y \star X$ is given by

$$Y \star X(x, y, z, h) = \begin{cases} 1 & \text{if } |x| = |y| = 1, z = h = \emptyset, \\ 1 & \text{if } |x| = |h| = 1, y = z = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we have that $y \star x = |Y \star X| = xy + 2xh$.

2. The functor $Z \star X$ is given by

$$Z \star X(x, y, z, h) = \begin{cases} 1 & \text{if } |x| = |z| = 1, y = h = \emptyset, \\ -1 & \text{if } |y| = |h| = 1, x = z = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we have that $z \star x = |Z \star X| = xz - yh$.

3. The functor $Z \star Y$ is given by

$$Z \star Y(x, y, z, h) = \begin{cases} 1 & \text{if } |b| = |c| = 1, a = d = \emptyset, \\ 2 & \text{if } |b| = |d| = 1, a = c = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we have that $z \star y = |Z \star Y| = yz + 2yh$.

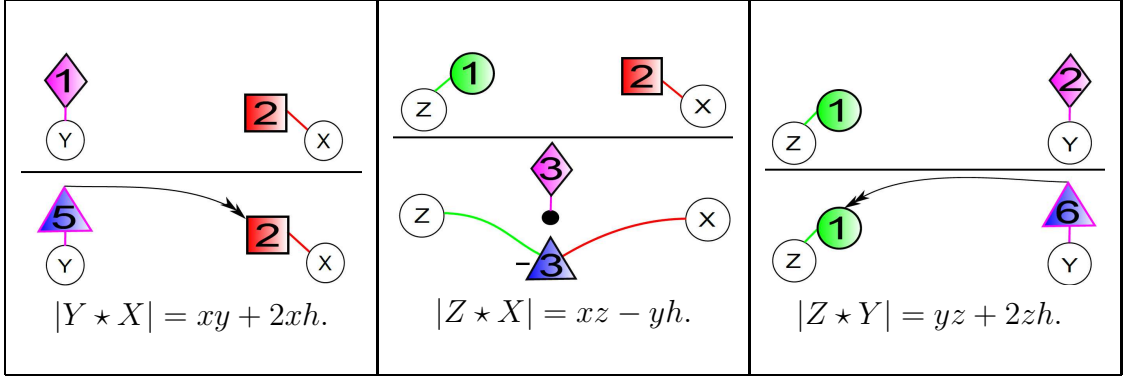


Figure 5: Graphical representation for the defining identities of $\hat{U}_h(\mathfrak{sl}_2)$.

Proof. The reader should have Figure 5 in mind as we develop our arguments.

1. By the definition of the singular species there must be exactly one y -element attached to the Y -blob, and exactly one x -element attached to the X -blob. This can happen in two ways: either we originally have the required elements, or we had an h -element and a x -element. The h -element falls into the third block of the decomposition of h and thus becomes the needed y -element. In this case we must also consider the maps from h_5 to the disjoint union of two copies of the x -set, yielding the required factor of 2.
2. Again we have two cases: either we have an x -element and a z -element, or we have an h -element and a y -element. The h -element falls in block number 3 and fills the place of the z and x elements needed. The y -element goes to the third block, its attached to the black disk yielding a factor of 1. The contribution of a graph with $|h| \geq 2$ is equal to 0. Indeed, the only active blocks for the partition of h are h_3 and h_4 , and we know that $|h_3| \leq 1$ (otherwise the applications of the functor Z yields a 0 factor.) Now if $|h| \geq 3$, then $|h_4| = |h| - |h_3| \geq 2 > |h_3|$, a contradiction since we know that $|h_4| \leq |h_3|$. If $|h| = 2$ and $|h_3| = |h_4| = 1$, then $|y_3| = 0$ and $(z_1 + x_2)_{y_3+h_4}^{h_4}$ gives rise to a factor of $(0 + 0)_{0+1}^1 = 0$.
3. There must be exactly one z -element attached to the Z -blob, and exactly one x -element attached to the X -blob. So either we are given the required elements, or we had an h -element and a x -element. The h -element necessarily falls into the block h_6 and thus becomes the required y -element. We must also consider the maps from h_6 to the disjoint union of two copies of the z -set, yielding the required factor of 2.

□

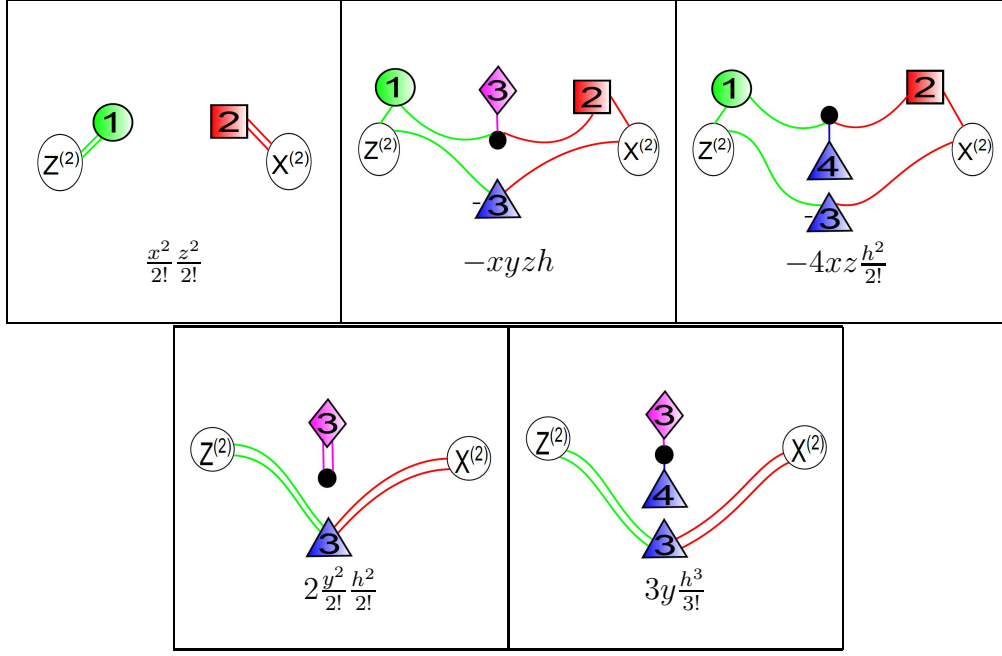


Figure 6: Graphical interpretation of the \star -product functor $\frac{Z^2}{2!} \star \frac{X^2}{2!}$.

4.2 Graphical representation of the identities from Lemma 2

In this subsection we study the graphical representation of the identities in Lemma 2. We begin with an example, namely, we consider the \star -product

$$\frac{Z^2}{2!} \star \frac{X^2}{2!}$$

for which we adopt the multiple-lines representation to be fully explicit. Figure 6 displays the various graphs that arise in this computations together with their associated algebraic counterpart. Our goal is to construct all graphs that can be built as in Figure 4 with $F = \frac{Z^2}{2!}$ and $G = \frac{X^2}{2!}$, proceeding in increasing order in the cardinality of h .

Proposition 10. The functor $\frac{Z^2}{2!} \star \frac{X^2}{2!} \in [\mathbb{B}^4, C]$ is such that

$$\left| \frac{Z^2}{2!} \star \frac{X^2}{2!}(x, y, z, h) \right| = \begin{cases} 1 & \text{if } |x| = |z| = 1, y = h = \emptyset, \\ -1 & \text{if } |x| = |y| = |z| = |h| = 1, \\ -4 & \text{if } |x| = |z| = 1, |h| = 2, y = \emptyset, \\ 2 & \text{if } |y| = |h| = 2, x = z = \emptyset, \\ 3 & \text{if } |y| = 1, |h| = 3, x = z = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we have that

$$\frac{z^2 x^2}{2! 2!} = \left| \frac{Z^2 X^2}{2! 2!} \right| = \frac{x^2 z^2}{2! 2!} - xyzh - 4xz \frac{h^2}{2!} + 2 \frac{y^2 h^2}{2! 2!} + 3y \frac{h^3}{3!}.$$

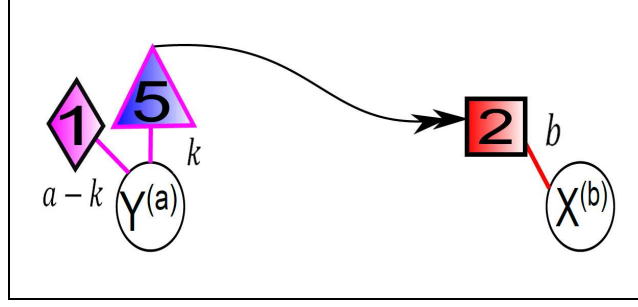


Figure 7: Graphical interpretation of the \star -product functor $\frac{Y^a}{a!} \star \frac{X^b}{b!}$.

Proof. The reader should have Figure 6 in mind as we develop our arguments.

- The only graph we can build with $|h| = 0$ is the one with two z -lines attached to the left blob, and two x -lines attached to the right blob.
- There is only one graph with $|h| = 1$. Indeed, the unique h -colored element must necessarily lie in the block h_3 , given rise a z -colored edge connected to the left blob, as well as a x -colored edge connected to the right blob. Since $|h_3| = 1$, then either $|y_3| = 1$ and $|h_4| = 0$, or $|y_3| = 0$ and $|h_4| = 1$. The later option is not allowed since we are assuming that $|h| = |h_3| = 1$. Thus we have that $(z_1 + x_2)_{y_3+h_4}^{h_4}$ gives rise to a factor of $(1 + 1)_1^0 = 1$.
- There are two cases with $|h| = 2$. Assume first that $|h_3| = |h_4| = 1$, then we obtain a factor of 2 accounting for the partitions h in two blocks. Also we have that $(z_1 + x_2)_{y_3+h_4}^{h_4}$ gives rise to a factor of $(1 + 1)_1^1 = 2$. Thus we obtain the desired factor of -4 .
- Next we assume that $|h| = |h_3| = 2$ which implies that $|h_4| = 0$ and $|y_3| = 2$. In this case $(z_1 + x_2)_{y_3+h_4}^{h_4}$ gives rise to a factor of $(0 + 0)_{2+0}^0 = 1$.
- Consider the case $|h| = 3$. We have that $|h_3| + |h_4| = 3$ and $|y_3| + |h_4| = |h_3|$. If $|h_3| = 2$ and $|y_3| = |h_4| = 1$, then $(z_1 + x_2)_{y_3+h_4}^{h_4}$ gives rise to a factor of $(0 + 0)_{1+1}^1 = (0)_2^1 = 0 + 1 = 1$.
- If $|h| \geq 5$, then $|h_4| = |h| - |h_3| \geq 3 > |h_3|$ a contradiction since we know that $|h_4| \leq |h_3|$. Thus there are no contribution to the product from such graphs.
- If $|h| = 4$, then we must have that $|h_3| = |h_4| = 2$ and $|y_3| = 0$. Therefore $(z_1 + x_2)_{y_3+h_4}^{h_4}$ gives rise to a factor of $(0 + 0)_{0+2}^2 = 0$.

□

Next we consider the general case.

Theorem 11. Consider the category $([\mathbb{B}^4, C], \star)$. For $a, b \in \mathbb{N}$ we have that:

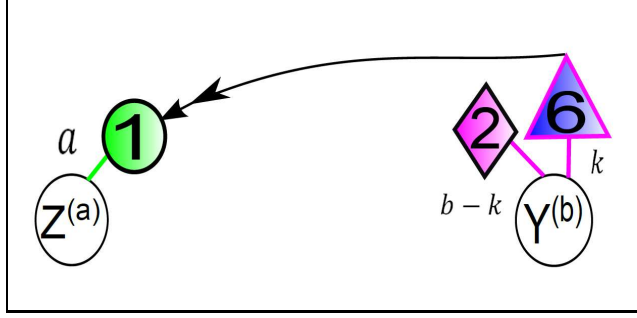


Figure 8: Graphical interpretation of the \star -product functor $\frac{Z^a}{a!} \star \frac{Y^b}{b!}$.

1. The functor $\frac{Y^a}{a!} \star \frac{X^b}{b!}$ is given by

$$\frac{Y^a}{a!} \star \frac{X^b}{b!}(x, y, z, h) = \begin{cases} [h, x \sqcup x] & \text{if } |x| = b, |y| + |h| = a, z = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we have that

$$\frac{y^a}{a!} \star \frac{x^b}{b!} = \left| \frac{Y^a}{a!} \star \frac{X^b}{b!} \right| = \sum_{k=0}^a (2b)^k \frac{x^b}{b!} \frac{y^{a-k}}{(a-k)!} \frac{h^k}{k!}.$$

2. The functor $\frac{Z^a}{a!} \star \frac{Y^b}{b!}$ is given by

$$\frac{Z^a}{a!} \star \frac{Y^b}{b!}(x, y, z, h) = \begin{cases} [h, z \sqcup z] & \text{if } x = \emptyset, |y| + |h| = b, |z| = a \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we have that

$$\frac{z^a}{a!} \star \frac{y^b}{b!} = \left| \frac{Z^a}{a!} \star \frac{Y^b}{b!} \right| = \sum_{k=0}^b (2a)^k \frac{y^{b-k}}{(b-k)!} \frac{z^a}{a!} \frac{h^k}{k!}.$$

3. The functor $\frac{Z^a}{a!} \star \frac{X^b}{b!}$ is such that

$$\frac{Z^a}{a!} \star \frac{X^b}{b!}(x, y, z, h) = \bigoplus_{w \sqcup v = h} (-1)^{|v|} \mathbb{L}(w) \otimes \mathbb{L}(y) \otimes (x \sqcup z)_v^w,$$

where in the sum above the following conditions must be satisfied

$$|v| \leq \min(a, b), \quad |x| + |v| = b, \quad |y| + |w| = |v|, \quad |z| + |v| = a.$$

Therefore we have that

$$\begin{aligned} \frac{z^a}{a!} \star \frac{x^b}{b!} &= \left| \frac{Z^a}{a!} \star \frac{X^b}{b!} \right| = \\ &= \sum_{0 \leq w \leq v \leq \min(a, b)} (-1)^v (v-w)! (v+w)_w (a+b-2v)_v^w \frac{x^{b-v}}{(b-v)!} \frac{y^{v-w}}{(v-w)!} \frac{z^{a-v}}{(a-v)!} \frac{h^{v+w}}{(v+w)!}. \end{aligned}$$

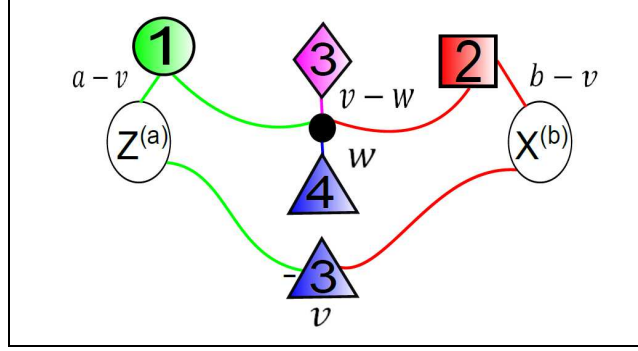


Figure 9: Graphical interpretation of the \star -product functor $\frac{Z^a}{a!} \star \frac{X^b}{b!}$.

Proof. The reader should have Figures 7, 8, 9 in mind as we go along the proof.

1. Clearly in this case the only non-empty block in the partition of h is h_5 . Therefore we have that $0 \leq |h| = |h_5| \leq a$. If $|h| = k$, then we still need $a - k$ elements of color y , and b elements of color x . The map from h_5 to $[b] \sqcup [b]$ gives rise to the factor $(2b)^k$.
2. Similarly in this case the only non-empty block in the partition of h is h_6 . Therefore we have that $0 \leq |h| = |h_6| \leq b$. If $|h| = k$, then we still need $b - k$ elements of color y , and a elements of color x . The map from h_6 to $[a] \sqcup [a]$ gives rise to the factor $(2a)^k$.
3. In this case h is partitioned in two blocks h_3 and h_4 with $0 \leq |h_4| \leq |h_3| \leq \min(a, b)$. Set $w = |h_4|$ and $v = |h_3|$. So we have that $|y_3| = v - w$. Thus we need an additional set with $a - v$ elements with color z , and another set with $b - v$ elements of colored x . Therefore $(z_1 + x_2)_{y_3+h_4}^{h_4}$ gives rise to a factor of $(a - v + b - v)_{v-w+w}^w = (a + b - 2v)_v^w$. \square

4.3 Graphs and the \star -product of the exponentiated variables

In this final subsection we study with graphical methods the \star -product of the exponentiated variables in $\widehat{U}_h(\mathfrak{sl}_2)$. First we recall the combinatorial meaning of the exponentiated variables. The functor E^X , similar constructions applied for the other variables, is given on a 4-tuple of finite sets $(a, b, c, d) \in \mathbb{B}^4$ by

$$E^X(a, b, c, d) = \begin{cases} 1 & \text{if } b = c = d = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

We have that $|E^X| = e^x$, and we can similarly define functors E^Y , E^Z and E^H such that $|E^Y| = e^y$, $|E^Z| = e^z$ and $|E^H| = e^h$.

Theorem 12. Consider the category $([\mathbb{B}^4, C], \star)$. We have that:

1. The functor $E^Y \star E^X$ is such that $E^Y \star E^X(x, y, z, h) = \emptyset$ if $z \neq \emptyset$, and otherwise it is given by

$$E^Y \star E^X(x, y, z, h) = [h, x \sqcup x],$$

and therefore

$$e^y \star e^x = |E^Y \star E^X| = e^{xe^{2h}} e^y.$$

2. The functor $E^Z \star E^Y$ is such that $E^Z \star E^Y(x, y, z, h) = \emptyset$ if $y \neq \emptyset$, and otherwise it is given by

$$E^Z \star E^Y(x, y, z, h) = [h, z \sqcup z],$$

and therefore

$$e^z \star e^y = |E^Z \star E^Y| = e^x e^{ze^{2h}}.$$

3. The functor $E^Z \star E^X$ is given by

$$E^Z \star E^X(x, y, z, h) = \bigoplus_{w \sqcup v = h} (-1)^{|v|} \mathbb{L}(w) \otimes \mathbb{L}(y) \otimes (x \sqcup z)_v^w,$$

where in the sum above the identity $|y| + |w| = |v|$ should hold. Therefore

$$e^z \star e^x = |E^Z \star E^X| = \sum_{a, c, w \leq v \in \mathbb{N}} (-1)^v (v - w)! (v + w)_w (a + c)_v^w \frac{x^a}{a!} \frac{y^{v-w}}{(v - w)!} \frac{z^c}{c!} \frac{h^{v+w}}{(v + w)!}.$$

Proof. The proof is similar to that of Theorem 11. Again the reader should have Figures 7, 8, 9 in mind, but replacing the application of divided powers functors by the applications of the corresponding exponentiated variables functors. Thus most of the restrictions on the cardinality of sets are lifted. Item 3 follows then directly. Let us show item 1. From the previous considerations we have that:

$$\begin{aligned} e^y \star e^x &= |E^Y \star E^X| = \sum_{a, b, c \in \mathbb{N}} (2a)^c \frac{x^a}{a!} \frac{y^b}{b!} \frac{h^c}{c!} = \sum_{a, c \in \mathbb{N}} \frac{x^a}{a!} \frac{(2ah)^c}{c!} e^y = \\ &= \sum_{a \in \mathbb{N}} \frac{x^a}{a!} e^{2ah} e^y = \sum_{a \in \mathbb{N}} \frac{(xe^{2h})^a}{a!} e^y = e^{xe^{2h}} e^y. \end{aligned}$$

□

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